Worcester County Mathematics League

Varsity Meet 2
December 2, 2015

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 1: Fractions, Decimals, and Percents

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. $\frac{1}{2}$ of $\frac{1}{3}$ is what fraction of $\frac{1}{2}+\frac{1}{3}$?

2. Lisa will make punch that is 25% fruit juice by adding pure fruit juice to a 2-liter mixture that is 10% fruit juice. How many liters of pure fruit juice does she need to add?

3. A used video game is marked down by 30% and you have a coupon allowing an additional 25% off the discount price. A sales tax of 5% is applied to video games after any discounts are applied. You paid \$43.96 for the game. What was the original sticker price for the game?

ANSWERS

(1 pt.) 1. ______________________

(2 pt.) 2. ______________________ liters

(3 pt.) 3. \$______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 2: Algebra I

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. Farmer Rick wants to build a fence enclosing a 128 square meter rectangular grazing area. If the length of the grazing area is four times the width, what is the perimeter of the enclosure?

2. Kim can paint $\frac{1}{3}$ of a fence in 2 hours. It takes Viktor 5 hours to paint $\frac{1}{2}$ of the fence. If they work together, how many minutes will it take for them to paint the entire fence?

3. Geppetto began building Pinocchio at 5:00 p.m. To the nearest second, when is the next time that the angle between the minute and hour hands will have the same measure as they did when Geppetto began?

ANSWERS

(1 pt.) 1. ______________________ meters

(2 pt.) 2. ______________________ minutes

(3 pt.) 3. _______ : _______ : _______ p.m.

WORCESTER COUNTY MATHEMATICS LEAGUE

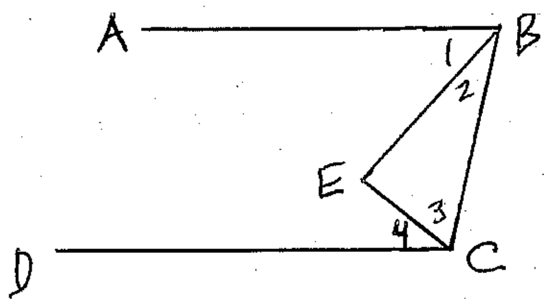
Varsity Meet 2 - December 2, 2015
Round 3: Parallel Lines and Polygons

All answers must be in simplest, exact form in the answer section.

NO CALCULATOR ALLOWED

1. The measure of each interior angle of a regular polygon is 90° less than eight times each exterior angle. Each of the polygon's sides is 6 inches long. Find the polygon's perimeter in inches.

2. $\overline{AB} \parallel \overline{CD}$, $m\angle 1 = 2m\angle 2$, $m\angle 3 = 3m\angle 4$, and $m\angle E = 80°$. Find $m\angle ABC$.



3. Polygon B has 5 more sides and 90 more diagonals than Polygon A. How many sides does polygon A have?

ANSWERS

(1 pt.) 1. ______________________ inches

(2 pt.) 2. ______________________ °

(3 pt.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 4: Sequences and Series

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. For what value(s) of x will $x+2$, $3x+1$, and $4x-5$, in this order, form an arithmetic sequence?

2. The first term of an infinite geometric series is 12. If S is the sum of the series, for what value(s) of S does the series converge?

3. The n^{th} term of a sequence is given by $t_n = At_{n-3} + Bt_{n-2} + Ct_{n-1}$. If $t_5 = 3t_1 + t_2 - t_3$ and $t_6 = -3t_1 + 5t_2$, determine the product of D, E, and F, where $t_8 = Dt_1 + Et_2 + Ft_3$.

ANSWERS

(1 pt.) 1. $x=$ ______________________

(2 pt.) 2. ______________________

(3 pt.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 5: Matrices and Systems of Equations

All answers must be in simplest, exact form in the answer section.

NO CALCULATOR ALLOWED

1. For $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 9 \\ 6 & 4 & 6 \end{bmatrix}$, determine $B - 2A$.

2. Find a if $a \geq 0$, $\begin{bmatrix} a & b \\ b & b \end{bmatrix} \times \begin{bmatrix} b & a \\ a & b \end{bmatrix} = \begin{bmatrix} d & d \\ d & d \end{bmatrix}$, and $\det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = 8$.

3. Solve over the set of real numbers: $\det \begin{bmatrix} 2x & 3x-2 & 0 \\ 3 & x & 1 \\ 1-x & 2 & x \end{bmatrix} = 40$

ANSWERS

(1 pt.) 1. $\begin{bmatrix} \quad \end{bmatrix}$

(2 pt.) 2. $a=$ ____________

(3 pt.) 3. $X=$ ____________

WORCESTER COUNTY MATHEMATICS LEAGU

Varsity Meet 2 – December 2, 2015
Team Round Answers

All answers must be in simplest, exact form.
All questions in this round are worth 2 points.

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________ cm^3

5. $______________________________

6. ______________________________

7. ______________________________ km/hr

8. ______________________________

9. ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Team Round

All answers must be in simplest, exact form in the answer section.
All questions in this round are worth 2 points.

1. If the sum of the first n terms of the sequence 2015, 2012, 2009, ... is 510545, find n.

2. Bird seed comes in 3-lb and 7-lb bags. Customers bought a total of 44 pounds of bird seed. How many bags of each size were sold? Write all possible answers as ordered pairs, where the first coordinate is the number of 3-kb bags and the second is the number of 7-lb bags.

3. If $\begin{bmatrix} 3 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} a & c \\ -b & a \\ c & b \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 4 & -3 \end{bmatrix}$, then what is the value of $a+b+c$?

4. A right rectangular prism has dimensions x cm, $(x+1)$ cm, and $(x+2)$ cm, and its total surface area is 100 cm^2. Find the volume of the prism to the nearest cm^3.

5. Ricardo wants to buy two pairs of jeans. One pair costs \$50.00. He has two sale coupons. The first states, "Buy one, get one 50% off, where the 50% is taken off the cheaper pair." The second says, "Take 15% off your entire purchase." The store only allows one coupon per purchase. If Ricardo saves more money by using the first coupon, what is the most Ricardo can expect to pay for the two pairs of jeans after the coupon is applied? (Assume that no sales tax is applied.)

6. Consider a parabola of the form $Ax^2 + Bx + C$. Determine the sum of the coefficients A, B, and C for the parabola that passes through the points $(2, 15)$, $(3, 13)$, and $(6, -17)$.

7. Sam walked 8 km into the country. He returned walking 5 km/hr slower. The total time for his round trip was 6 hours. How quickly did he walk on the return trip?

8. In $\triangle ABC$, median $\overline{AM}$ is such that $m\angle BAM = \frac{1}{2}m\angle MAC$ and $\overline{AM}$ is extended through M to D so that $\angle DBA$ is a right angle. Find the ratio of AD to AC.

9. A triangle has vertices $(-1, a)$, $(5, 2)$, and $(1, 8)$. For what value(s) of a will the area of the triangle be 28 units^2?

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015

ANSWERS

Round 1

1. $\frac{1}{5}$ or 0.2 (St. John's)
2. $\frac{2}{5}$ or 0.4 (Shepherd Hill)
3. \$79.75 (Northbridge)

Round 2

1. $40\sqrt{2}$ (Bromfield)
2. 225 (Holliston)
3. 5:54:33 (Worcester Academy)

Round 3

1. 72 (Algonquin)
2. 84 (Westborough)
3. 17 (Worcester Academy)

Round 4

1. -5 (Doherty)
2. $S > 6$ (Shepherd Hill) → Note: Any equivalent statement of inequality is acceptable.
   For example: $6 < S < \infty$ and $(6, \infty)$ are both acceptable.
3. -300 (Auburn)

Round 5

1. $\begin{bmatrix} -4 & -1 & 0 \\ -9 & -5 & 9 \\ 2 & -2 & 4 \end{bmatrix}$ (Westborough)
2. $2\sqrt{2}$ (Southbridge)
3. 6 (Bancroft)

Team Round

1. 1006 (Tantasqua)
2. (3, 5) and (10, 2) (West Boylston)
   *need both
3. 2 (AMSA Charter)
4. 66 (Worcester Academy)
5. 141.66 (Bromfield)
6. 13 (Doherty)
7. $\frac{5}{3}$ or $1\frac{2}{3}$ or 1.667 (Doherty)
8. 1:2 (Bartlett)
9. -3 or 25 (Shepherd Hill)
   *need both

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 1: Fractions, Decimals, and Percents

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. $\frac{1}{2}$ of $\frac{1}{3}$ is what fraction of $\frac{1}{2}+\frac{1}{3}$?
The statement must be translated into a mathematical expression and simplified.

$$\frac{1}{2}\times\frac{1}{3}=x(\frac{1}{2}+\frac{1}{3})$$
$$\frac{1}{6}=x(\frac{3}{6}+\frac{2}{6})$$
$$\frac{1}{6}=\frac{5x}{6}$$
$$1=5x$$
$$x=\frac{1}{5}$$

2. Lisa will make punch that is 25% fruit juice by adding pure fruit juice to a 2-liter mixture that is 10% pure fruit juice. How many liters of pure fruit juice does she need to add?

Let x be the amount of pure fruit juice Lisa adds to the juice mixture. The statement can be expressed mathematically as $100x+10(2)=25(x+2)$ and needs simplification.

$$100x+20=25x+50$$

$$75x=30$$

$x=\frac{30}{75}=\frac{2}{5}=0.4$ liters. We have that Lisa must add .4 liters of pure fruit juice.

3. A used video game is marked down by 30% and you have a coupon allowing an additional 25% off the discount price. A sales tax of 5% on video games after any discounts are applied. You paid \$43.96 for the game. What was the original sticker price for the game?

Let x represent the original sticker price of the game. Then the above scenario may be expressed mathematically as $43.96=\cancel{(.3)(.25)(1.05)x}$ $(.7)(.75)(1.05)x$

$$43.96=\frac{4396}{100}=\frac{2198}{50}=\frac{1099}{25}=(.7)(.75)(1.05)x=(\frac{7}{10})(\frac{3}{4})(\frac{21}{20})x=\frac{441}{800}$$
$$x=\frac{1099}{25}\times\frac{800}{441}=\frac{1099}{1}\times\frac{32}{441}$$

Long division yields $x=79.746$. The original sticker price of the game is \$79.75.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 2: Algebra I

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. Farmer Rick wants to build a fence enclosing a 128 square meter rectangular grazing area. If the length of the grazing area is four times the width, what is the perimeter of the enclosure?

Denote the width of the fenced in area by w and the length by l.

$l = 4w$ and $l \times w = 128$

Simple substitution for l into the second equation in terms of w yields $(4w)(w) = 4w^2 = 128$.

$w^2 = 32$

$w = \sqrt{32} = \sqrt{2 \times 16} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

Since measurements cannot be negative, the width of the fenced in area is $4\sqrt{2}$ meters. Substituting $w = 4\sqrt{2}$ into the equation $l = 4w$ gives $l = 4(4\sqrt{2}) = 16\sqrt{2}$. The total amount of fencing required to construct the enclosure is $2(4\sqrt{2} + 16\sqrt{2}) = 40\sqrt{2}$ meters.

2. Kim can paint $\frac{1}{3}$ of a fence in 2 hours. It takes Viktor 5 hours to paint $\frac{1}{2}$ of the fence. If they work together, how many minutes will it take for them to paint the entire fence?

The rate at which Kim can paint the fence is $\frac{\frac{1}{3}}{120}\frac{fence}{minutes}$. Viktor can paint the fence at a rate of $\frac{\frac{1}{2}}{300}\frac{fence}{minutes}$. Let x be the number of minutes it will take for Kim and Viktor working together to paint the entire fence. Then

$$(\frac{\frac{1}{3}}{120})\frac{fence}{minutes}x + (\frac{\frac{1}{2}}{300})\frac{fence}{minutes}x = 1.$$

$$(\frac{\frac{1}{3}}{120})\frac{fence}{minutes}x + (\frac{\frac{1}{2}}{300})\frac{fence}{minutes}x = (\frac{1}{360})\frac{fence}{minutes}x + (\frac{1}{600})\frac{fence}{minutes}x = (\frac{1}{6})\frac{fence}{hours}x + (\frac{1}{10})\frac{fence}{hours}x$$

$$(\frac{1}{6})x + (\frac{1}{10})x = 1$$

$$5x + 3x = 8x = 30$$

$$x = \frac{30}{8} = \frac{15}{4} \text{ hours}$$

$$x = (\frac{15}{4}) \text{ hours } (60\frac{minutes}{hour}) = 225 \text{ minutes}$$

It will take Kim and Viktor 225 minutes to paint the entire fence.

3. Geppetto began building Pinocchio at 5:00 p.m. To the nearest second, when is the next time that the angle between the minute and hour hands will have the same measure as they did when Geppetto began?

The angle between the minute and hour hands of the clock when Geppetto began working on Pinocchio is 150°. The rate at which the minute hand moves is $\frac{6°}{minute}$, and the hour hand moves at a rate of

$\frac{30°}{hour}$. Since the minute hand moves at a faster rate than the hour hand, the angle between the hands will only decrease until the minute hand surpasses the hour hand. The equation that represents the number of minutes it will take for the hands of the clock to form the same 150° angle is given by $\frac{6°}{minute}x - (\frac{30°}{hour} + 150)x = 150$, where x is the number of minutes.

$$\frac{6°}{minute}x - [(\frac{30°}{hour})x + 150] = \frac{6°}{minute}x - [(\frac{30°}{60minutes})x + 150] = 150$$

Since the equation is now in minutes and degrees and all terms have the same units, units can be removed from the equation.

$$\frac{6}{1}x - [(\frac{30}{60})x + 150] = \frac{6}{1}x - [(\frac{1}{2})x + 150] = 12x - x - 300 = 300$$

$$12x - x - 300 = 11x - 300 = 300$$

$$11x = 600$$

$$x = \frac{600}{11} = 54\frac{6}{11} \text{ minutes}$$

It will take $54\frac{6}{11}$ minutes, or 54 minutes and 33 seconds, for the hands to form the same angle as when Geppetto began building Pinocchio. So the next time this angle will be formed is at 5:54 and 33 seconds, or 5:54:33 p.m.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 3: Parallel Lines and Polygons

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. The measure of each interior angle of a regular polygon is 90° less than eight times each exterior angle. Each of the polygon's sides is 6 inches long. Find the polygon's perimeter.

Let x represent the measure of the exterior angle of the regular polygon. Then

$180 - x = 8x - 90 \rightarrow 270 = 9x \rightarrow 30 = x$

The sum of the exterior angles of the polygon is $360°$, so $n \cdot 30 = 360$, where n is the number of sides of the regular polygon. $n = 12$, so the perimeter of the regular polygon is $12 \cdot 6 = 72$ in.

2. $\overline{AB} \parallel \overline{CD}$, $m\angle 1 = 2m\angle 2$, $m\angle 3 = 3m\angle 4$, and $m\angle E = 80°$. Find $m\angle ABC$.

Let $m\angle 2 = a$ and $m\angle 4 = b$. Then $m\angle 1 = 2a$ and $m\angle 3 = 3b$.

The sum of the measures of the angles of a triangle is $180°$.
So $a + 3b + 80 = 180 \rightarrow a + 3b = 100$.

Same-side interior angles of parallel lines cut by a transversal are supplementary.
So $m\angle ABC + m\angle DCB = 180 \rightarrow 3a + 4b = 180$.

Solving the system, we find $b = 24$, $a = 28$, and $m\angle ABC = 2(28) + 28 = 84°$.

3. Polygon B has 5 more sides and 90 more diagonals than Polygon A. How many sides does polygon A have?

If polygon A has n sides, it has $\frac{n(n-3)}{2}$ diagonals.
Polygon B has $(n + 5)$ sides and $\frac{(n+5)[(n+5)-3]}{2} = \frac{(n+5)(n+2)}{2}$ diagonals.

So $\frac{n(n-3)}{2} + 90 = \frac{(n+5)(n+2)}{2}$ and $n^2 - 3n + 180 = n^2 + 7n + 10$.
Solving for n, we find $10n = 170$, and $n = 17$.

So Polygon A has 17 sides.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 4: Sequences and Series

All answers must be in simplest, exact form in the answer section.
NO CALCULATOR ALLOWED

1. For what value(s) of x will $x+2$, $3x+1$, and $4x-5$, in this order, form an arithmetic sequence?

The difference between the 2nd and 1st terms is the same as the difference between the 3rd and 2nd terms:

$$(3x+1)-(x+2)=(4x-5)-(3x+1) \rightarrow 2x-1=x-6 \rightarrow x=-5$$

The value of x for which the given terms form an arithmetic sequence is -5.

2. The first term of an infinite geometric series is 12. If S is the sum of the series, for what value(s) of S does the series converge?

The series converges when the sum, as given by the standard formula $S=\dfrac{a}{1-r}$ exists and $-1<r<1$.

$S=\frac{12}{1-r}$. When $-1<r<1$, $0<1-r<2$ and $6<S<\infty$. So the series converges for any sum greater than 6.

3. The n^{th} term of a sequence is given by $t_n=At_{n-3}+Bt_{n-2}+Ct_{n-1}$. If $t_5=3t_1+t_2-t_3$ and $t_6=-3t_1+5t_2$, determine the product of D, E, and F, where $t_8=Dt_1+Et_2+Ft_3$.

First find A, B, and C.

$$t_4=At_1+Bt_2+Ct_3$$

$$\begin{aligned} t_5=At_2+Bt_3+Ct_4=At_2+Bt_3+C(At_1+Bt_2+Ct_3) &= ACt_1+(A+BC)t_2+(B+C^2)t_3 \\ &= 3t_1+t_2-t_3 \end{aligned}$$

So $AC=3, A+BC=1,$ and $B+C^2=-1$.

$$\begin{aligned} t_6=At_3+Bt_4+Ct_5 &= At_3+B(At_1+Bt_2+Ct_3)+C(3t_1+t_2-t_3) \\ &= (AB+3C)t_1+(B^2+C)t_2+(A+BC-C)t_3 \\ &= -3t_1+5t_2 \end{aligned}$$

So $AB+3C=-3, B^2+C=5,$ and $A+BC-C=0$.

Since $A+BC=1, (A+BC)-C=1-C=0$. So $C=1$. $AC=3,$ so $A=3$. $A+BC=1,$ so $3+B=1$ and $B=-2$.

Now $t_4=3t_1-2t_2+t_3$.

So $t_7 = 3t_4 - 2t_5 + t_6 = 3(3t_1 - 2t_2 + t_3) - 2(3t_1 + t_2 - t_3) + (-3t_1 + 5t_2) = -3t_2 + 5t_3$

and $t_8 = 3t_5 - 2t_6 + t_7 = 3(3t_1 + t_2 - t_3) - 2(-3t_1 + 5t_2) + (-3t_2 + 5t_3) = 15t_1 - 10t_2 + 2t_3$.

So $D = 15, E = -10$, and $F = 2$. The required product $DEF = 15(-10)(2) = -300$.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Round 5: Matrices and Systems of Equations

All answers must be in simplest, exact form in the answer section.

NO CALCULATOR ALLOWED

1. For $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 9 \\ 6 & 4 & 6 \end{bmatrix}$, determine $B - 2A$.

$$B - 2A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 9 \\ 6 & 4 & 6 \end{bmatrix} - 2 \times \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 9 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3(2) & 1(2) & 0(2) \\ 6(2) & 4(2) & 0(2) \\ 2(2) & 3(2) & 1(2) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 9 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 2 & 0 \\ 12 & 8 & 0 \\ 4 & 6 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} (2-6) & (1-2) & (0-0) \\ (3-12) & (3-8) & (9-0) \\ (6-4) & (4-6) & (6-2) \end{bmatrix} = \begin{bmatrix} -4 & -1 & 0 \\ -9 & -5 & 9 \\ 2 & -2 & 4 \end{bmatrix}$$

2. Find a if $a \geq 0$, $\begin{bmatrix} a & b \\ b & b \end{bmatrix} \times \begin{bmatrix} b & a \\ a & b \end{bmatrix} = \begin{bmatrix} d & d \\ d & d \end{bmatrix}$, and $\det A \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = 8$.

The determinant of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is $ab - 0(0) = ab = 8$.

$$\begin{bmatrix} a & b \\ b & b \end{bmatrix} \times \begin{bmatrix} b & a \\ a & b \end{bmatrix} = \begin{bmatrix} (ab + ba) & (a^2 + b^2) \\ (b^2 + ba) & (ba + b^2) \end{bmatrix} = \begin{bmatrix} (2ab) & (a^2 + b^2) \\ (b^2 + ba) & (ba + b^2) \end{bmatrix} = \begin{bmatrix} d & d \\ d & d \end{bmatrix}$$

$2ab = a^2 + b^2 = ab + b^2 = d$

The two middle equations above give $a^2 = ab$. It is also known that $ab = 8$.

Equating the two gives $a^2 = 8$, or $a = 2\sqrt{2}$. It is stipulated that $a \geq 0$, so $a = 2\sqrt{2}$.

3. Solve over the set of real numbers: $\det \begin{bmatrix} 2x & 3x-2 & 0 \\ 3 & x & 1 \\ 1-x & 2 & x \end{bmatrix} = 40$

$$\det \begin{bmatrix} 2x & 3x-2 & 0 \\ 3 & x & 1 \\ 1-x & 2 & x \end{bmatrix} = 2x^3 + 3x - 3x^2 - 2 + 2x - 9x^2 + 6x - 4x = 2x^3 - 12x^2 + 7x - 2 = 40$$

$2x^3 - 12x^2 + 7x - 42 = 0$

This cubic is easily factored.

$2x^3 - 12x^2 + 7x - 42 = 2x^2(x-6) + 7(x-6) = (2x^2 + 7)(x - 6)$

The only real root to this cubic is $x = 6$.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - December 2, 2015
Team Round

All answers must be in simplest, exact form in the answer section.
All questions in this round are worth 2 points.

1. If the sum of the first n terms of the sequence 2015, 2012, 2009, ... is 510545, find n.

This is an arithmetic sequence with a common difference of -3. The nth term of this sequence is $a_n = 2015 - 3(n-1) = 2015 - 3n + 3$. The sum of the first n terms of an arithmetic sequence is given by $S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2015 + 2015 - 3n + 3) = \frac{4033}{2}n - \frac{3}{2}n^2$. Setting this expression equal to the given sum of 510545, we have $510545 = \frac{4033}{2}n - \frac{3}{2}n^2$.
So $1021090 = 4033n - 3n^2$, and $3n^2 - 4033n + 1021090 = 0$.
Apply the quadratic formula: $n = \frac{-(-4033)\pm\sqrt{(-4033)^2-4(3)(1021090)}}{2(3)}$
So $n = \frac{4033+\sqrt{4033^2-12(1021090)}}{6} = 1006$ or $n = \frac{4033-\sqrt{4033^2-12(1021090)}}{6} = \frac{1015}{3}$.
Since n must be an integer, n is 1006.

2. Bird seed comes in 3-lb and 7-lb bags. Customers bought a total of 44 pounds of bird seed. How many bags of each size were sold? Write all possible answers as ordered pairs, where the first coordinate is the number of 3-lb bags and the second is the number of 7-lb bags.

The answer here comes from exhausting all options. There is only one equation and two variables, but the possibilities are limited enough to be discovered in a reasonable amount of time. The equation is $3x + 7y = 44$. The answers are $(3,5)$ and $(10,2)$

3. If $\begin{bmatrix} 3 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} a & c \\ -b & a \\ c & b \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 4 & -3 \end{bmatrix}$, then what is the value of $a+b+c$?

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} a & c \\ -b & a \\ c & b \end{bmatrix} = \begin{bmatrix} 3a+c & 3c+b \\ -2a-3b & -2c+3a \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 4 & -3 \end{bmatrix}.$$

$$\begin{cases} 3a+c=6 \\ 3c+b=7 \\ -2a-3b=4 \\ -2c+3a=-3 \end{cases}$$

Subtracting the final equation from the first gives $3c = 9$, or $c = 3$. Substituting this value into the first and second equations give $3a + 3 = 6$, or $a = 1$, and $9 + b = 7$, so $b = -2$. The value of $a + b + c$ is $a + b + c = 1 - 2 + 3 = 2$.

4. A right rectangular prism has dimensions x cm, $(x + 1)$ cm, and $(x + 2)$ cm, and its total surface area is 100 cm^2. Find the volume of the prism to the nearest cm^3.

The equation for the total surface area is the value of each of the six surfaces' areas combined. Thus,

$$2(x)(x+1) + 2(x)(x+2) + 2(x+1)(x+2) = 2(x^2 + x + x^2 + 2x + x^2 + 3x + 2) = 2(3x^2 + 6x + 2) = 100$$

$$3x^2 + 6x + 2 = 50$$

$$3x^2 + 6x - 48 = 0$$

$$x^2 + 2x - 16 = 0$$

Apply the quadratic formula: $x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-16)}}{2(1)} = \frac{-2 \pm \sqrt{68}}{2} = \frac{-2 \pm 2\sqrt{17}}{2} = -1 \pm \sqrt{17}$
Since x must be positive, $x = -1 + \sqrt{17}$. The dimensions of the prism are $-1 + \sqrt{17}, \sqrt{17}$, and $1 + \sqrt{17}$, and the volume is the product of these dimensions: $(-1 + \sqrt{17})(\sqrt{17})(1 + \sqrt{17}) =$ 65.97 cm^3. To the nearest cm^3, the volume is 66 cm^3.

5. Ricardo wants to buy two pairs of jeans. One pair costs \$50.00. He has two sale coupons. The first states, "Buy one, get one 50% off, where the 50% is taken off the cheaper pair." The second says, "Take 15% off your entire purchase." The store only allows one coupon per purchase. If Ricardo saves more money by using the first coupon, what is the most Ricardo can expect to pay for the two pairs of jeans after the coupon is applied? (Assume that no sales tax is applied.)

Let x be the cost of the second pair of jeans.

If $x <$ \$50.00, then $50 + 0.5x < 0.85(x + 50)$.
So $50 + 0.5x < 0.85x + 42.5$, and $7.5 < 0.35x \rightarrow 21.429 < x < 50 \rightarrow \$21.43 \leq x \leq \$49.99$.
The total purchase price (after the coupon is applied) is between \$60.72 and \$75, inclusive.

If $x >$ \$50.00, then $x + 0.5(50) < 0.85(x + 50)$.
So $x + 25 < 0.85x + 42.5$, and $0.15x < 17.5 \rightarrow 50 < x < 116.667 \rightarrow \$50.01 \leq x \leq \$116.66$.
The total purchase price (after the coupon is applied) is between \$75.01 and \$141.66, inclusive.

So the most Ricardo can expect to pay for the two pairs of jeans is \$141.66.

6. Consider a parabola of the form $y = Ax^2 + Bx + C$. Determine the sum of the coefficients A, B, and C for the parabola that passes through the points (2, 15), (3, 13), and (6, –17).

The system of equations formed by inserting each point into the standard form quadratic equation is

$$\begin{cases} 4A + 2B + C = 15 \\ 9A + 3B + C = 13 \\ 36A + 6B + C = -17 \end{cases}.$$

Subtracting the first equation from the second and third equations gives, respectively:

$$\begin{cases} 5A + B = -2 \\ 32A + 4B = -32 \end{cases}.$$

Multiplying the first of these equations by 4 and subtracting it from the second yields $12A = -24$, or $A = -2$. Substituting this value into the expression derived above and solving for B yields $5(-2) + B = -2$, and $B = -2 + 10 = 8$. Substituting the values for A and B into any of the original equations gives the value for C. For simplicity, substitution into the first equation yields $4(-2) + 2(8) + C = -8 + 16 + C = C + 8 = 15$, and $C = 15 - 8 = 7$. The sum $A + B + C$ is $-2 + 8 + 7 = 13$.

7. Sam walked 8 km into the country. He returned walking 5 km/hr slower. The total time for his round trip was 6 hours. How quickly did he walk on the return trip?

Let x denote the rate at which Sam walked into the country (in km/h). Then

$$\frac{\text{distance walked into country}}{\text{rate walked into country}} + \frac{\text{distance walked on return trip}}{\text{rate walked on return trip}} = \text{total time}$$

So $\frac{8}{x} + \frac{8}{x-5} = 6$ and...

$8(x-5) + 8x = 6x(x-5)$

$8x - 40 + 8x = 6x^2 - 30x$

$0 = 6x^2 - 46x + 40$

$0 = 3x^2 - 23x + 20$

$0 = (3x - 20)(x - 1)$

So $x = \frac{20}{3}$ or $x = 1$. If $x = 1$, Sam's return rate would be –4 km/h. Since this is impossible, $x = \frac{20}{3}$, and Sam walked $\left(\frac{20}{3} - 5\right)$ km/h $= \frac{5}{3}$ km/h on his return trip.

8. In $\triangle ABC$, median $\overline{AM}$ is such that $m\angle BAM = \frac{1}{2}m\angle MAC$ and $\overline{AM}$ is extended through M to D so that $\angle DBA$ is a right angle. Find the ratio of AD to AC.

Let $m\angle BAM = x$; then $m\angle MAC = 2x$. Extend $\overline{AD}$ to point P so that $ACPB$ is a parallelogram. Opposite sides of a parallelogram are congruent, so $BP = AC$. The diagonals bisect each other, so $AM = MP$. Let T be the midpoint of $\overline{AD}$ making $\overline{BT}$ the median of right $\triangle ABC$. Then $BT = \frac{1}{2}AD$, or $BT = AT$. Therefore $m\angle TBA = x$. $\angle BTP$, an exterior angle of isosceles $\triangle BTA$, has measure $2x$. Alternate interior angles of parallel lines cut by a transversal are congruent, so $m\angle CAP = m\angle BPA = 2x$. So $\triangle TBP$ is isosceles with $BT = BP$. Since $BT = \frac{1}{2}AD$ and $BT = BP = AC$, $AC = \frac{1}{2}AD$, and the ratio of AC to AD is 1:2.

Note that this can be seen quite clearly when $\triangle ABC$ is a 30-60-90 triangle. Let $m\angle BAC = 90$, $m\angle ABC = 30$, and $m\angle ACB = 60$. Then $\triangle AMC$ and $\triangle DMB$ are congruent equilateral triangles, so $AD = 2AC$.

9. A triangle has vertices $(-1, a)$, $(5, 2)$, and $(1, 8)$. For what value(s) of a will the area of the triangle be 28 units^2?

The area of a triangle with vertices (a, b), (c, d), and (e, f) is the absolute value of $\frac{1}{2}\det\begin{bmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{bmatrix}$.

Then $\pm 28 = \frac{1}{2}\det\begin{bmatrix} -1 & a & 1 \\ 5 & 2 & 1 \\ 1 & 8 & 1 \end{bmatrix}$.

$\pm 56 = -2 + a + 40 + 8 - 5a - 2 = 44 - 4a$

So $12 = -4a$ or $-100 = -4a$, and $a = -3$ or $a = 25$.